Since

$$\nu_1^2 = \max_v \frac{(v,\ v)}{(Bv,\ v)}$$

then $\nu_1 \leqslant \nu_1$.

If $R \geqslant 0$, then taking into account that $(Bv,\ v) > 0$, we obtain the inequality $\alpha_1 \geqslant 0$ from the definition of α_1. The proposed assertion is proved completely. Let us note that the sign in front of the root in (14) has been selected so that the equality $q_1' = q_1 = i\nu_1$ would hold for $R=0$.

N o t e 2. The following theorem is proved by the reasoning presented.

T h e o r e m. Let B be a positive definite operator in the Hilbert space H, which has a completely continuous inverse operator B^{-1}. Let R be a linear operator, where $B^{-1}R$ is a completely continuous operator. Let $\nu_1^2 \geqslant \nu_2^2 \geqslant \ldots > 0$ be the eigennumbers of the operator B^{-1}. Then the imaginary part of any eigennumber of the operator $q^2B + qR + I$ is not greater than ν_1, but if the operator R is nonnegative, then the real parts of the eigennumbers of the operator $q^2B + qR + I$ are nonpositive.

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OSCILLATIONS OF SYSTEMS WITH TIME - DEPENDENT PARAMETERS

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There are several approximate methods [1-4] available for solving practical problems having to do with oscillatory systems whose parameters vary with time. The procedure for analyzing such systems proposed in the present paper is based on the analogy between parametric and forced oscillations in a certain nominal oscillator with parameters chosen in a certain special way. Our approach, which closely resembles the idea behind the WKB (Wentzel-Kramers-Brillouin) method [3-5], provides increased opportunities for constructing effective approximate solutions of problems of the indicated class.

1. We begin by considering the following linear second-order differential equation to which many problems of applied dynamics can be reduced [1 and 5]:

$$q + 2n\,(t)q + k^2\,(t)q = F\,(t) \tag{1.1}$$

The Euler substitution reduces Eq. (1.1) to the form

$$y^{\cdot\cdot} + p^2\,(t)\,y = Q\,(t), \qquad (p^2 = k^2 - n^2 - n) \tag{1.2}$$

$$y = q \exp\left[\int_0^t n\,(t)\,dt\right], \qquad Q = F \exp\left[\int_0^t n\,(t)\,dt\right]$$

The solution of the homogeneous equation corresponding to Eq. (1.2) is obtainable in the form

$$y^* = B\ (t) \cos \Phi\ (t) \tag{1.3}$$

Here

$$y^{*\cdot\cdot} = (B^{\cdot\cdot} - B\Phi^{\cdot}) \cos \Phi - (2B^{\cdot}\Phi^{\cdot} + B\Phi^{\cdot\cdot}) \sin\Phi \tag{1.4}$$

Form (1. 3) enables us to introduce an additional condition, which we choose in such a way that the coefficient of sin Φ in (1. 4) vanishes,

$$2B^{\cdot}\Omega + B\Omega^{\cdot} = 0 \qquad (\Omega = d\Phi/dt) \tag{1.5}$$

Considering (1. 5) as a differential equation with separable variables, we obtain

$$B = A\ \sqrt{\Omega_0/\Omega} \qquad (\Omega_0 = \Omega\ (0)) \tag{1.6}$$

Here A is an arbitrary constant. Hence,

$$y^* = A\ \sqrt{\Omega_0 / \Omega}\ \cos\left(\int_0^t \Omega\, dt + \gamma\right) \tag{1.7}$$

where γ is the initial phase.

Substituting (1. 7) into the original homogeneous equation, we obtain

$$z^{\cdot\cdot} - 0.5z^{\cdot 2} + 2\Omega_*^2 e^{2z} = 2p^2(t) \qquad (z = \ln \Omega / \Omega_*) \tag{1.8}$$

Here Ω_* is an arbitrary parameter having the dimensions of frequency.

This equation corresponds to some single-mass nonlinear oscillator with alternating-sign "damping"; the role of the external perturbing force in this case is played by the function $2p^2\ (t)$. The variable z can be regarded here merely as the analog of some elastic strain, so that we shall refer to the oscillator as "nominal". If $p^2\ (t)$ is a slowly varying function, then the forced oscillations of the oscillator practically coincide with the static strain produced by the applied perturbing force ; here $\Omega \approx p$. It is interesting to note that in this particular case expression (1. 7) has the form of the solution obtained for differential equation (1. 2) by means of a first-order WKB approximation [4] ; it is also the form of the principal term of an asymptotic representation obtained by the method of reduction to the so-called L-diagonal form [6].

The particular solution y^{**} of nonhomogeneous equation (1. 2) obtained by the method of variation of arbitrary constants on the basis of solution (1. 7) is of the form

$$y^{**} = \frac{1}{\sqrt{\Omega(t)}} \int_0^t Q_1(u) \sin\left[\Phi\ (t) - \Phi\ (u)\right] du \qquad \left(Q_1(u) = \frac{Q\ (u)}{\sqrt{\Omega\ (u)}}\right) \tag{1.9}$$

Making use of (1. 7) and (1. 9), we can express the general solution of Eq. (1. 1) as

$$q = v \exp\left[-\int_0^t n\ (t)\ dt - 0.5\ (z - z_0)\right] \tag{1.10}$$

Here

$$v = y \sqrt{\frac{\Omega}{\Omega_0}} = v_0 \cos \Phi + v_0' \sin \Phi + \int_{\Phi_0}^{\Phi} F_1(\psi) \sin (\Phi - \psi)\ d\psi \tag{1.11}$$

$$\left(v_0 = v\ (\Phi_0) = y_0, \quad v_0' = \left(\frac{dv}{d\Phi}(\Phi_0) = \frac{y_0^{\cdot}}{\Omega_0} + 0.5\ \frac{\Omega_0^{\cdot}}{\Omega_0^2}\right)\left(F_1(\psi) = \frac{Q\ (\psi)}{\sqrt{\Omega_0 \Omega^3 (\psi)}}\right)\right.$$

We note that $v\ (\Phi)$ is the solution of the differential equation $v'' + v = F_1\ (\Phi)$ in which the role of "time" is played by the function Φ.

2. Let us consider the effect of a jump in $p^2\ (t)$ in order to investigate the dynamic properties of the nominal oscillator. Let

$$p^2(t) = p_0^2 + [p_1^2 - p_0^2]\,\eta\,(t - t_0)$$

where $\eta\,(t - t_0)$ is a unit step function ($\eta = 0$ for $t < t_0$, $\eta = 1$ for $t > t_0$).

Converting to the dimensionless time $\tau = 2\Omega_* t$, we can rewrite Eq. (1.8) as follows:

$$z'' - 0.5z'^2 + 0.5e^{2z} = 0.5\nu^2(\tau) \qquad (2.1)$$

$$\left(z' = \frac{dz}{d\tau} = \frac{z^{\cdot}}{2\Omega_*}, \quad z'' = \frac{d^2z}{d\tau^2} = \frac{z^{\cdot\cdot}}{4\Omega_*^2}, \quad \nu = \frac{p}{\Omega_*}\right)$$

Equation (2.1) has an exact solution for $\nu = \text{const}$. Setting $(z')^2 = x$, we reduce (2.1) to a first-order differential equation with constant coefficients,

$$dx \,/\, dz - x = \nu^2 - e^2 z \qquad (2.2)$$

The solution of this equation is reducible to the form

$$z' = \pm\sqrt{he^{z-z_0} - (\nu^2 + e^{2z})} \qquad (2.3)$$

$$\tau - \tau_0 = \int_{z_0}^{z} \frac{dz}{z'} = \frac{1}{\nu}\left|\operatorname{Arc}\sin \frac{he^{z-z_0} - 2\nu^2}{e^z\sqrt{h^2 - 4\nu^2}}\right|_{z_0}^{z} \qquad (2.4)$$

$$\left(h = z_0'^2 + e^{2z_0} + \nu^2, \quad z_0 = \ln\frac{\Omega_0}{\Omega_*} \quad z_0' = \frac{\Omega_0^{\cdot}}{2\Omega_*\Omega_0}\right)$$

We can show that $h \geqslant 2\nu$ in all cases.

In our case $\Omega_0 = p_0$, $\Omega_0^{\cdot} = 0$, so that the initial conditions can be written as $z_0 = \ln \nu_0$, $z_0' = 0$.

Setting $\Omega_* = p_1$ ($\nu_1 = 1$), we used (2.3) to construct phase trajectories for several values of ν_0 (Fig. 1). These trajectories are closed curves symmetric with respect to the axis of abscissas; they intersect the latter at the two points $z = \pm \ln \nu_0$ equidistant from the origin. We have also plotted some transitional curves N_1N_i ($i = 2 \div 5$) in the phase plane; these curves correspond to a linear decrease of $\nu^2(\tau)$ from $\nu_0^2 = 10$ to $\nu^2(\tau_*) = \nu_1^2 = 1$.

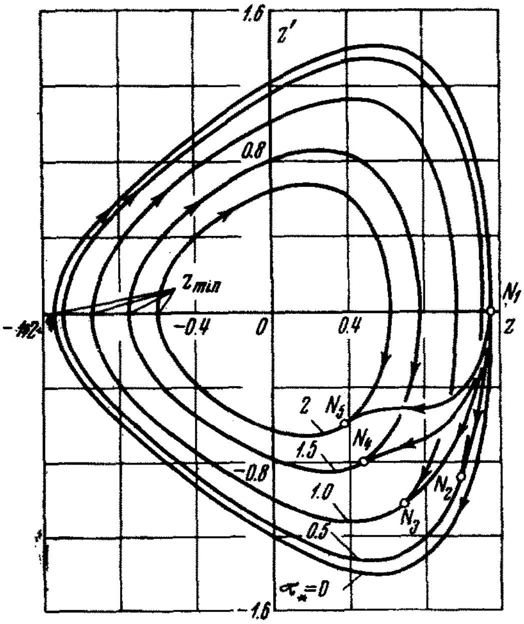

Fig. 1

Analyzing relation (1.6), we can express the maximum of the variable amplitude $B(t)$ as

$$B_{\max} = A \exp\left[-0.5\,(z_{\min} - z_0)\right] \qquad (2.5)$$

It is easy to show on the basis of (2.4) that the oscillation period τ_z of the nominal oscillator corresponding to traversal of the entire phase trajectory contour is equal to 2π; converting to the variable t, we find that the period corresponding to the frequency $2p_1$ is $T_z = \pi/p_1$.

3. First let us use the above method to consider some typical problems

which will enable us to compare our results with known solutions. Let

$$p^2 = p_*^2 (1 - 2\varepsilon \cos \omega t) \tag{3.1}$$

Here (1.2) assumes the form of the Mathieu equation.

Applying the method of harmonic linearization to nonlinear differential equation (2.1) and setting $\Omega_* = p_*$, we obtain

$$z = a_0 + a \cos \omega t \tag{3.2}$$

Here a, a_0 can be found from the relations

$$a = \frac{\varepsilon \varkappa^2 (a)}{|\varkappa^2 (a) - (\omega / 2p_*)^2|} \frac{a I_0 (2a)}{I_1 (2a)}, \quad a_0 = \frac{1}{2} \ln \frac{p_*^2 + 0.125\omega^2}{p_*^2 I_0 (2a)} \tag{3.3}$$

$$\left(\varkappa^2 (a) = \frac{I_1 (2a)}{a [I_0 (2a) - 0.5a I_1 (2a)]}, \quad I_k (2a) = i^{-k} J_k (2ai), \quad i = \sqrt{-1}\right)$$

where J_k is a Bessel function of the first kind of the imaginary argument $2ai$ [7]. Analysis shows that for $a \leqslant 1$ the fundamental proper frequency of the nominal oscillator assumes values $1 \leqslant \varkappa \leqslant 1.035$.

For example, let us determine the free oscillations for $\varepsilon = 0.1$, $\omega = 1.71\ p_*$, $n = 0{,}02\ p_*$. Making use of the tables of the functions I_0, I_1 appearing in [7], we find from (3.3) that $a = 0.4$, $a_0 = 0{,}08$. Hence, from formula (1.10) we have

$$q = A \exp [-nt + 0.5a (1 - \cos \omega t)] \cos [\Omega_* e^{a_0} \int_0^t \exp (a \cos \omega t)\, dt + \gamma] \approx A \exp [-0.01\, \tau + 0.2 (1 - \cos 0.855\, \tau)] \cos [0.54 (\tau + 0.467 \sin 0.855\tau)]$$

The curve of $q(\tau)$ for $A = 1$ and $\gamma = 0$ appears in Fig. 2.

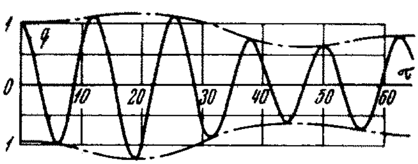

Fig. 2

Figure 3 illustrates the development of resonance oscillations of the nominal oscillator for $\omega = 2p_*$ and $\omega = p_*$, and also the parametric resonances associated with these states. It is interesting to note that at the principal parametric resonance ($\omega = 2p_*$) the rise of oscillations of the nominal oscillator is very closely approximated by the relation

$$z = \varepsilon p_* t \sin 2p_* t \tag{3.4}$$

which is a particular solution of the differential equation

$$z^{\cdot\cdot} + 4p_*^2 z = -4\varepsilon p_*^2 \cos 2p_* t$$

obtained from (1.8) by linearization for small z.

We infer from (1.10), (1.11) that for $Q = 0$ the function q is bounded in the parametric resonance zones if

$$\int_0^T n(t)\, dt > 0.5\, |\Delta z| \quad \left(T = \frac{2\pi}{p_*}\right) \tag{3.5}$$

Here Δz is the difference between the minima of the function z separated by the period T. For $n = \text{const}$ with allowance for (3.4) and (3.5) we have

$$n > 0.5\ \varepsilon p_* \tag{3.6}$$

This agrees with the familiar result obtained in analyzing the truncated Hill determinant [1].

Let us consider another typical case of periodic variation of the function $p^2(t)$ represented as a "rectangular sine function" with a pulsation equal to μp_*^2.

Since $p^2(t)$ is a piecewise-constant function, we can use the exact solution of Eq. (2.1) in the form (2.3). Let there be an extremum $z = z_1$ in the interval of the first half-wave ($p^2 = p_1^2$). The value of this extremum can then be determined from the following quadratic equation obtained from (2.3) by setting $z' = 0$:

$$e^{2z_1} - (\nu_1^2 e^{-z_0} + e^{z_0}) e^{z_1} + \nu_1^2 = 0 \quad (3.7)$$

This extremum is associated with the root

$$e^{z_1} = \nu_1^2 e^{-z_0} \quad (3.8)$$

If we set $\Omega_* = p_*$, then $\nu_0 = 1$ and $z_0 = 0$; moreover, $z_1 = \ln \nu_1^2 = \ln (1 - \mu)$.

Let the switchings from one segment to the next occur at the instants of attainment of the extrema (Fig. 4). This case (in which the durations of the upper and lower "half-waves" are strictly speaking different) is most hazardous from the standpoint of parameteric excitation. Making use of relations of the type (3.8), we can readily show that the minima z_1, z_2, z_3, etc., form an arithmetic series whose difference is equal to $\ln [(1 - \mu) / (1 + \mu)]$.

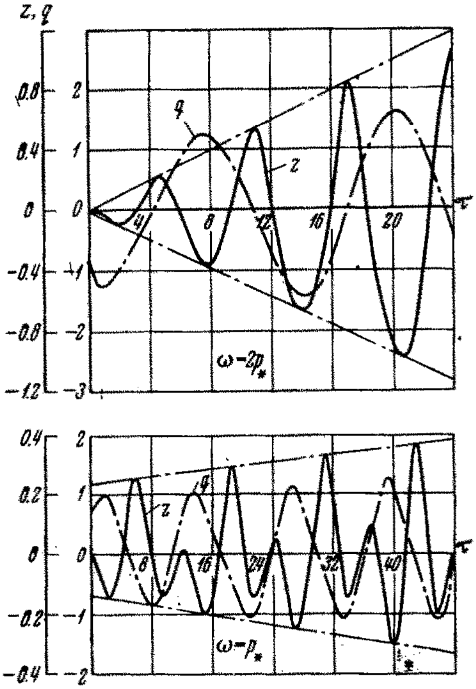

Fig. 3

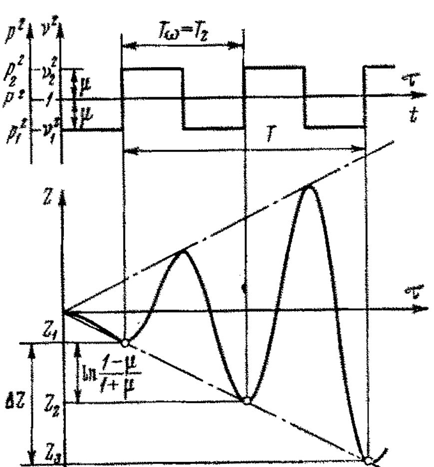

Fig. 4

Recalling that in this case the average period of characteristic oscillations (i.e. $T = 2\pi / p_*$), the oscillation period T_z of the nominal oscillator, and the period T_ω of the rectangular sine function are related as $T = 2T_z = 2T_\omega$ (see Sect. 2), we find that $\Delta z = 2 \ln [(1 - \mu) / (1 + \mu)]$. Substituting Δz into condition (3.5), we obtain

$$\lambda = \int_0^T n(t)\,dt > \left| \ln \frac{1-\mu}{1+\mu} \right| \quad (3.9)$$

Here λ is the average value of the logarithmic decrement per period.

It can be shown that similar conditions which build up the oscillation of z arise not only in the above case, but also for $T_\omega = jT_z = 0.5j$ where $j = 1, 3, 5, \ldots$. We can then write condition (3.9) in the more general form

$$\lambda > \frac{1}{j} \left| \ln \frac{1-\mu}{1+\mu} \right| = \frac{2}{j}\left(\mu + \frac{\mu^3}{3} + \frac{\mu^5}{5} + \ldots\right) \quad (3.10)$$

The critical frequencies $\omega = 2\pi / T_\omega$ corresponding to the principal parametric resonances lie near the values $2p_* / j$. If we retain the first term of the series in (3.10) for $j = 1$, then the result also coincides with the approximate condition obtained in analyz-

$$\| a_{ij}(t) \| \{y_i^{\cdot\cdot}\} + \| c_{ij}(t) \| \{y_i\} = 0 \qquad (i, j = 1, \ldots, s) \qquad (5.1)$$

where $\| a_{ij}(t) \|$, $\| c_{ij}(t) \|$ are the matrices of the inertial and elastic coefficients; s is the number of degrees of freedom.

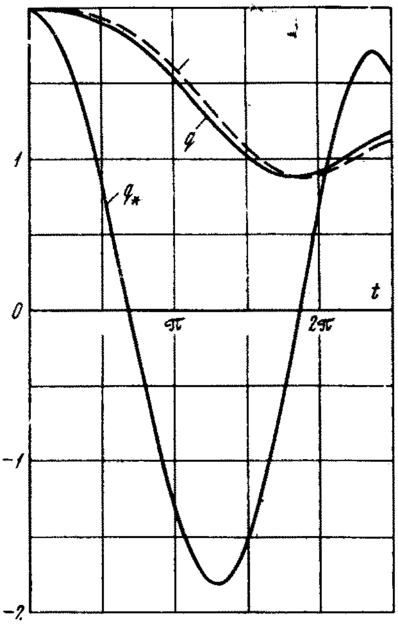

Fig. 5

We assume that for any t the formal frequency equation obtained for "frozen" coefficients has s simple roots. The solution y_i can be obtained in series form,

$$y_i = \sum_{r=1}^{s} B_{ir}(t) \cos \Phi_r(t) \qquad (5.2)$$

Satisfying conditions of the form (1.5), we can write

$$y_i = \sum_{r=1}^{s} A_{ir} \left(\frac{\Omega_r(0)}{\Omega_r(t)}\right)^{1/2} \cos\left[\int_0^t \Omega_r(t)\,dt + \gamma_r\right] =$$

$$= A_{ii} \sum_{r=1}^{s} \alpha_{ir}(t) \left(\frac{\Omega_r(0)}{\Omega_r(t)}\right)^{1/2} \cos\left[\int_0^t \Omega_r(t)\,dt + \gamma_r\right]$$

$$\left(\alpha_{ir} = \frac{A_{ir}}{A_{rr}}\right) \qquad (5.3)$$

Here the function $\alpha_{ir}(t)$ characterizes the variable oscillation mode.

If $a_{ij}(t)$ and $c_{ij}(t)$ are certain piecewise-constant functions, then, substituting (5.3) into (5.1) and ignoring the trivial solution $y_i \equiv 0$, we carry out some transformations to obtain

$$z_r^{\cdot\cdot} - 0.5 z_r^{\cdot 2} + 2\Omega_{*r}^{2} e^{2z_r} = 2p_r^2(t) \quad (z_r = \ln(\Omega_r/\Omega_{*r})) \quad (r = 1, \ldots, s) \qquad (5.4)$$

where Ω_{*r} is an arbitrary parameter.

The function $p_r^2(t)$ is defined as the root of the equation

$$\det \| c_{ij}(t) - a_{ij}(t) p^2(t) p^2 \| = 0 \qquad (5.5)$$

Similarly, $\Omega_r(t)$ and $p_r(t)$ are also related in a broader class of problems if

$$\left|\frac{\alpha_{ir}}{\alpha_{ir} p_r}\right| \ll 1, \quad \left|\frac{\alpha_{ir}}{\alpha_{ir} p_r^2}\right| \ll 1 \qquad (5.6)$$

This practically common case occurs when the functions $\alpha_{ir}(t)$ vary slowly or have a small pulsation depth.

Equation (5.4) is of the same form as (1.8) and corresponds to s independent nominal oscillators which vary only in their "perturbations". Hence under these assumptions the series of results formulated in Sects, 2 and 3 can be used for analyzing system (5.1).

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STEADYSTATE MOTIONS IN AUTONOMOUS SYSTEMS WITH A DEVIATING ARGUMENT

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A rotating-phase autonomous system with a deviating argument is investigated. A scheme of successive approximations for the exact solution over an infinite time interval is constructed; sufficient conditions for the existence of a steadystate solution are derived. Such systems occur frequently in the theory of nonlinear vibrational-rotational motions in systems whose parameters vary within a narrow range.

Let us construct the stationary, i.e. steadystate, solutions of a real system of the form

$$dE / dt = \varepsilon f(E, E_\tau, \psi, \psi_\tau, \varepsilon) \qquad (E_\tau = E(t-\tau))$$
$$d\psi / dt = \omega(E, E_\tau) + \varepsilon F(E, E_\tau, \psi, \psi_\tau, \varepsilon) \qquad (\psi_\tau = \psi(t-\tau)) \qquad (1)$$

Here $t \in (-\infty, \infty)$ is the time, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ a small parameter, E a vector variable whose values lie in some neighborhood of the point E_0^*, $\psi \in (-\infty, \infty)$ the scalar phase, and $\tau \in (-\infty, \infty)$ a constant.

We can construct the solution by the method of successive approximations [1], making use of the fact that if system (1) has a solution $E(t), \psi(t)$ for all t, then it also has a family of solutions $E(t+\theta)$, $\psi(t+\theta)$, where θ is an arbitrary constant. The value of the phase ψ can therefore be chosen arbitrarily for some instant t_0. For example, we can set it equal to zero in order to simplify our expressions. To avoid secular terms in system (1) we introduce the new independent variable s such that

$$t - t_0 = s(1 + \varepsilon h), \qquad \tau = \varphi(1 + \varepsilon h)$$

This yields the system

$$dE / ds = \varepsilon(1 + \varepsilon h) f(E, E_\varphi, \psi, \psi_\varphi, \varepsilon)$$
$$d\psi / ds = (1 + \varepsilon h)[\omega(E, E_\varphi) + \varepsilon F(E, E_\varphi, \psi, \psi_\varphi, \varepsilon)]$$

where h is some constant which we choose in such a way that the solution of the perturbed system in s has the "unperturbed" period T_0.

Assuming that the functions f, ω have partial derivatives with respect to all their arguments and that these derivatives together with F satisfy the Lipschitz condition in the above domain, we make the substitutions

$$E = E_0 + \varepsilon x, \quad \psi = \omega_0 s + \theta + \varepsilon y \qquad (E_0, \theta = \text{const})$$

to obtain the system